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The dynamical Casimir effect for two oscillating mirrors in 3D

Cem Yuce and Zalihe Ozcakmakli

Department of Physics, Anadolu University, Eskisehir, Turkey

E-mail: cyuce@anadolu.edu.tr

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Abstract
The generation of photons in a three-dimensional rectangular cavity with two moving boundaries is studied by using the multiple scale analysis (MSA). It is shown that the number of photons are enhanced for the cavity whose walls oscillate symmetrically with respect to the center of the cavity. The non-stationary Casimir effect is also discussed for the cavity which oscillates as a whole.

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1. Introduction

When two perfectly conducting plates are placed close to each other, the attractive force appears between the parallel conductors due to the vacuum fluctuations as predicted a long time ago by Casimir [1]. The corresponding vacuum energies and forces are static. Now let us assume that the right boundary depends on time. In this case, the length of the cavity changes in time, $L = L(t)$. The most evident manifestation of dynamic behavior is the dependence of the force on time. In an adiabatic situation, the time-dependent modified force is given by [2]

$$F = \frac{\pi \hbar c}{24 L(t)}. \tag{1}$$

More strikingly, when the right plate is in motion, it was theoretically predicted that photons are generated in the empty cavity, because of the instability of the vacuum state of the electromagnetic field in the presence of time-dependent boundary conditions [3–12]. A number of virtual photons from the vacuum are converted into real photons. This phenomenon is known as the dynamical Casimir effect or motion-induced radiation. However, there has been no experimental verification for this effect up to now because of the technical difficulties. There are a few proposed experiments for the detection of photons [13–15]. As it was discussed in the literature, the best way to observe this effect is to vibrate one of the walls with one of the resonant field frequencies. A one-dimensional cavity with two perfectly parallel reflecting walls, one of which is motionless and the other oscillating with a mechanical frequency equal to a multiple of the fundamental optical resonance frequency of the static cavity, has been used as a simple model to study the dynamical Casimir effect [4–6, 12]

$$L_1(t) = 0, \qquad L_2(t) = L(1 + \epsilon \sin \Omega t), \tag{2}$$

where the constant Ω is the external frequency, ϵ is a small parameter, the constant L is the initial length of the cavity and $L_1(t)$ and $L_2(t)$ are the positions of the right and the left walls at time t, respectively. The cavity is motionless initially and at some instant one mirror starts to oscillate resonantly with a tiny amplitude.

Calculating the number of generated photons is a difficult task since one has to solve the wave equation with the time-dependent boundary conditions. A lot of techniques have been developed to deal with the problem. For example, averaging over fast oscillations [16, 17], multiple scale analysis [18], the rotating wave approximation [19], and numerical techniques [20] are applied to the dynamical Casimir problem.

The case of cavities with two moving mirrors was studied by a few authors [21–28]. Compared to the situation with a single oscillating mirror, it was found that radiation is resonantly enhanced when the cavity oscillates as a whole, with its mechanical length kept constant and when the cavity oscillates symmetrically with respect to the cavity center. The radiation emitted by two oscillating walls in one dimension was studied by Dalvit and Mazzitelli using the renormalization group method [23], by Dodonov using the method known in the theory of parametrically excited systems [25] and by Lambrecht *et al* using the scattering approach [28].

In this paper, we will study two moving boundary problems for a rectangular cavity resonator in 3D. We will use multiple scale analysis (MSA) which provides us with a solution valid for a period of time longer than that of the perturbative case, which is not suitable for this problem since it breaks down after a small time because of the resonance terms. We will investigate the two cases. In the first configuration, we will consider a rectangular cavity resonator whose right and left walls in the x-direction move in exactly the same way,

$$L_1(t) = \epsilon L \sin \Omega t, \qquad L_2(t) = L(1 + \epsilon \sin \Omega t). \tag{3}$$

Initially, the length of the oscillating cavity is given by $L_2(t) - L_1(t) = L$. As time goes on, the length of the cavity is not changed. This type of motion corresponds to the cavity oscillating as a whole.

Second, we will consider a rectangular cavity resonator whose right and left walls in the x-direction move opposite to each other,

$$L_1(t) = -\epsilon L \sin \Omega t, \qquad L_2(t) = L(1 + \epsilon \sin \Omega t). \tag{4}$$

We will show that the number of generated photons are enhanced for this system with the parametric resonance case, in which the frequency of the wall is twice the frequency of some unperturbed mode, say $\Omega = 2\omega_k$.

This paper is organized as follows. Section 2 studies the field quantization in the case of moving boundaries in one dimension. Section 3 reviews and applies the multiple scale analysis. Section 4 studies the dynamical Casimir effect for a three-dimensional cavity. Section 5 considers the cavity whose walls are oscillating symmetrically with respect to the center of the cavity. Finally, the last section discusses the enhancement of the generated photon number.

2. Field quantization with fixed length

Consider a one-dimensional cavity formed by two perfect conductors. The right and the left walls oscillate according to the formula given by (3). The cavity oscillates as a whole. The field operator in the Heisenberg representation $\Phi(x,t)$ obeys the wave equation $(c=1)$

$$\frac{\partial^2\Phi}{\partial t^2}-\frac{\partial^2\Phi}{\partial x^2}=0. \tag{5}$$

The boundary conditions are given by

$$\Phi(L_1,t)=\Phi(L_2,t)=0, \tag{6}$$

which describes the moving boundary problem in field theory. The problem looks simple, since the wave equation is the same whether the boundaries are moving or not. However, the moving boundary conditions render the equation unsolvable by the usual means. Although the solution of the wave equation is easy to find and well known, finding the exact solution of the same problem endowed with the time-dependent boundary conditions is very difficult and not known except for some special cases.

To solve the problem, we will transform the moving boundary conditions to the fixed boundary conditions. Let us introduce a coordinate transformation as

$$q(t)=\frac{x-L_1(t)}{L}. \tag{7}$$

As a result, the new time-independent boundary conditions for $\Phi(q,t)$ read

$$\Phi(q=0,t)=\Phi(q=1,t)=0. \tag{8}$$

With this coordinate transformation, the wave equation (5) is changed. Let us find the new form of the wave equation. Under the coordinate transformation (7), the derivative operators transform as

$$\begin{aligned}\partial_t^2 &\to \partial_t^2+\frac{\dot{L}_1{}^2}{L^2}\partial_q^2-2\frac{\dot{L}_1}{L}\partial_t\partial q-\frac{\ddot{L}_1}{L}\partial_q,\\ \partial_x^2 &\to \frac{1}{L^2}\partial_q^2.\end{aligned} \tag{9}$$

Here we use the notations $\partial_t\equiv\frac{\partial}{\partial t}$, $\partial_q\equiv\frac{\partial}{\partial q}$. By using the explicit forms of $L_1(t)$ and $L_2(t)$ (3), the transformation of the time derivative operator up to the first order of ϵ can be approximated as

$$\partial_t^2\approx\partial_t^2-\epsilon(2\Omega\ \cos\Omega t\,\partial_t\partial q-\Omega^2\ \sin\Omega t\,\partial_q). \tag{10}$$

Substituting these into the wave equation, we get

$$\frac{1}{L^2}\frac{\partial^2\Phi}{\partial q^2}=\frac{\partial^2\Phi}{\partial t^2}-\epsilon\left(\Omega\cos\Omega t\frac{\partial^2\Phi}{\partial q\partial t}-\Omega^2\sin\Omega t\frac{\partial\Phi}{\partial q}\right). \tag{11}$$

In the right-hand side of the equation, the term in the parenthesis is due to the time-dependent boundary conditions. Now, we do not have to deal with the time-dependent boundary conditions. However, we are left with a new equation.

Let us now solve the above equation subject to the fixed boundary conditions (8). The field operator can be expanded as

$$\Phi(q,t)=\sum_k\big(b_k\Psi_k(q,t)+b_k^\dagger\Psi_k^\star(q,t)\big), \tag{12}$$

where $b_k^\dagger$ and b_k are the creation and the annihilation operators, respectively and $\Psi_k(q,t)$ is the corresponding mode function. We will follow the approach given in [29–31] to find the explicit form of functions $\Psi_k(x,t)$. For an arbitrary moment of time, the mode function satisfying the boundary conditions is expanded as

$$\Psi_k(q, t > 0) = \sum_n a_n^k(t)\sin(n\pi q). \tag{13}$$

Let us substitute it into equation (11) in order to find $a_n^k(t)$. Then, multiply the resulting equation with $\sin(m\pi q)$ and integrate over q from 0 to 1. If we use the orthogonality relations, we get an infinite set of coupled differential equations for a_n^k after some algebra

$$\ddot{a}_m^k + \omega_m^2 a_m^k = \epsilon\left(4\Omega\cos\Omega t\sum_{n\neq m} g_{nm}\dot{a}_n^k - 2\Omega^2\sin\Omega t\sum_{n\neq m} g_{nm}a_n^k\right), \tag{14}$$

where $\omega_m = m\pi/L$ and the antisymmetric coefficient is given by $g_{nm} = \frac{mn(1-(-1)^{m+n})}{m^2-n^2}$ for $m \neq n$.

In the following section, we will solve the above equation. To do this, we prefer to use the multiple scale analysis method.

3. Multiple scale analysis (MSA)

Conventional weak-coupling perturbation theory suffers from problems that arise from resonant terms in the perturbation series. The effects of the resonance could be insignificant on short time scales but become important on long time scales. Perturbation methods generally break down after a small time whenever there is a resonance that leads to what are called secular terms. In equation (14), this happens for those particular values of external frequency Ω such that there is a resonant coupling with the eigenfrequencies of the static cavity. To avoid such problems, we will use multiple-scale analysis (MSA), a powerful and sophisticated perturbative method valid for longer times. Multiple-scale perturbation theory provides a good description of our system.

The trick is to introduce a new variable $\tau = \epsilon t$. This variable is called the slow time because it does not become significant in the small time. The functional dependence of a_m^k on t and ϵ is not disjoint because it depends on the combination of ϵt as well as on the individual t and ϵ. The time variables t and τ are treated independently in MSA. Thus, in place of $a_m^k(t)$, we write $a_m^k(t, \epsilon t)$. Let us expand a_m^k in the form of a power series in ϵ

$$a_m^k(t) = a_m^{k(0)}(t,\tau) + \epsilon a_m^{k(1)}(t,\tau) + \epsilon^2 a_m^{k(2)}(t,\tau) + \dots. \tag{15}$$

To this end, we change the independent variable in the original equation from t to τ. Using the chain rule, we have $\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial\tau}$.

Up to the first order of ϵ, the derivatives with respect to the time scale t are given by

$$\dot{a}_m^k = \partial_t a_m^{k(0)} + \epsilon\left(\partial_\tau a_m^{k(0)} + \partial_t a_m^{k(1)}\right) \qquad \ddot{a}_m^k = \partial_t^2 a_m^{k(0)} + \epsilon\left(2\partial_\tau\partial_t a_m^{k(0)} + \partial_t^2 a_m^{k(1)}\right), \tag{16}$$

where a dot denotes time derivation with respect to t as usual. Let us substitute these into equation (14). Then, we see that our original ordinary differential equation is replaced by a partial differential equation. It may appear that the problem has been complicated. But, as will be seen below, there are many advantages of this method. To zeroth order in ϵ, we get a well-known equation in physics,

$$\ddot{a}_m^{k(0)} + \omega_m^2 a_m^{k(0)} = 0. \tag{17}$$

To first order in ϵ, we obtain the following equation:

$$\ddot{a}_m^{k(1)} + \omega_m^2 a_m^{k(1)} = -2\partial_t\partial_\tau a_m^{k(0)} + 4\Omega\cos(\Omega t)\sum_{n\neq m} g_{nm}\dot{a}_n^{k(0)} - 2\Omega^2\sin(\Omega t)\sum_{n\neq m} g_{nm}a_n^{k(0)}. \tag{18}$$

The solution of the former one can be found easily

$$a_m^{k(0)}(t,\tau) = A_m^k(\tau)\,\mathrm{e}^{-\mathrm{i}\omega_m t} + B_m^k(\tau)\,\mathrm{e}^{\mathrm{i}\omega_m t}. \tag{19}$$

Note that $A_m^k(\tau)$ and $B_m^k(\tau)$ are not constants but functions of the slow time scales τ. The initial conditions are given by

$$A_m^k(\tau=0) = \frac{1}{\sqrt{2\omega_k}}\delta_{m,k} \qquad B_m^k(\tau=0) = 0. \tag{20}$$

Let us now solve equation (18). We look for the oscillations of the cavity that could enhance the number of generated photons by means of resonance effects for some specific external frequencies. To enhance the number of photons, let us now assume the resonance condition, $\Omega = p\pi/L$, where $p = 1, 2, \ldots$. It is well known that, in the resonance conditions, the number of generated photons grows very much in time.

Let us substitute the zeroth-order solution (19) into the right-hand side of equation (18) and then use the following relations: $2\mathrm{i}\sin\Omega t = (\mathrm{e}^{\mathrm{i}\Omega t} - \mathrm{e}^{-\mathrm{i}\Omega t})$, $2\cos\Omega t = (\mathrm{e}^{\mathrm{i}\Omega t} + \mathrm{e}^{-\mathrm{i}\Omega t})$. It can be seen that the right-hand side contains terms that produce secular terms. For a uniform expansion, these secular terms must vanish. In other words, any term with $\mathrm{e}^{\pm\mathrm{i}\omega_m t}$ on the right-hand side must vanish. If not, these terms would be in resonance with the left-hand side term and secularities would appear. After imposing the requirement that no term $\mathrm{e}^{-\mathrm{i}\omega_m t}$ appear, we get

$$\partial_\tau A_m^k + G^-_{p+m,m}A^k_{p+m} - G^+_{m-p,m}A^k_{m-p} - G^-_{p-m,m}B^k_{p-m} = 0, \tag{21}$$

where $G_{i,j}$ is defined as

$$G^{\mp}_{i,j} = \frac{\Omega \mp 2\omega_i}{2\omega_j}\,\Omega g_{ij}. \tag{22}$$

In the similar way, the fact that no secularities should arise from the term with $\mathrm{e}^{\mathrm{i}\omega_m t}$ leads to

$$-\partial_\tau B_m^k - G^-_{p+m,m}B^k_{p+m} + G^+_{m-p,m}B^k_{m-p} + G^-_{p-m,m}A^k_{p-m} = 0. \tag{23}$$

To this end, let us give the formula for the number of generated photons [7, 32]

$$\langle N_n\rangle = \sum_k 2\omega_n\left|B_n^k\right|^2. \tag{24}$$

We will now analyze the solutions of equations (21), (23) for a given p.

3.1. Analysis of solution

As a special case, let us study the above equations when $m = p$. In this case, equations (21), (23) are reduced to the following simple ones:

$$\partial_\tau A_p^k + G^-_{2p,p}A^k_{2p} = 0 \tag{25}$$

$$\partial_\tau B_p^k + G^-_{2p,p}B^k_{2p} = 0. \tag{26}$$

To find the solution, we should also find the corresponding differential equations for A_{2p}, B_{2p}. Substituting $m = 2p$ in (21), (23) gives

$$\partial_\tau A_{2p}^k + G^-_{3p,2p}A^k_{3p} - G^+_{p,2p}A^k_p = 0 \tag{27}$$

$$\partial_\tau B_{2p}^k + G^-_{3p,p}B^k_{3p} - G^+_{p,2p}B^k_p = 0. \tag{28}$$

Note that since $p - m \to p - 2p < 0$, the term with B^k_{p-m} vanishes. As can be seen easily, we need the equations A^k_{3p} and B^k_{3p} to solve these differential equations. In fact, there is no upper cutoff. However, these equations are not coupled equations. A_m^k and B_m^k are not coupled to each other. So, we say that no photons with $\omega_p, \omega_{2p}, \ldots$ are generated for a given external frequency $\Omega = \pi p/L$. But this does not mean that photons are not generated. For example, photons with ω_{p+1} are produced. To calculate the number of photons in this mode, we should solve (21), (23) when $m = p+1$. As was seen above, we are left with infinitely many equations since there is no cutoff. This is due to the fact that the spectrum of a one-dimensional cavity is equidistant. Intermode coupling produces resonant creation in the other modes. So, the spectrum does not have an upper frequency cutoff.

To sum up, photons are produced resonantly in all modes except $m = p, 2p, \ldots$. In the following section, we will study the dynamical Casimir effect for a three-dimensional cavity whose spectrum is not equidistant.

4. The dynamical Casimir effect in 3D

So far, we have restricted ourselves to the one-dimensional case. We will now study the dynamical Casimir effect for the three-dimensional geometries. We will show that multiple scale analysis works very well to calculate the number of generated photons.

Let us first define our problem. Consider a rectangular cavity resonator with perfectly conducting walls. Initially the three of the them are placed at $x = 0$, $y = 0$ and $z = 0$ while the other three walls are at $x = L_x$, $y = L_y$ and $z = L_z$. At time $t = 0$, the cavity starts oscillating in the x-direction as a whole. The positions of the six walls at any time are given by

$$L_{1x}(t) = \epsilon L_x \sin \Omega t, \qquad L_{2x}(t) = L_x(1 + \epsilon \sin \Omega t), \qquad L_{1y}(t) = 0,$$
$$L_{2y}(t) = L_y, \qquad L_{1z}(t) = 0, \qquad L_{2z}(t) = L_z, \tag{29}$$

where the constant Ω is the frequency of the oscillation and ϵ is a small parameter. The volume of the cavity resonator is constant in time and given by $V = L_x L_y L_z$.

As in the one-dimensional case, we will study with the coordinates transformed from the fixed ones to the moving ones, $(x, y, z) \rightarrow (q, y, z)$. The relation between q and x is given by $q = (x - L_{1_x})/L_x$.

We will first study the dynamical Casimir effect for the scalar field and then for the vector field.

4.1. *The scalar field in 3D*

In this section, we will apply MSA to the scalar field in three dimensions. The cavity resonator oscillates in the x-direction (29). So, the scalar field operator subject to the following boundary conditions:

$$\begin{aligned}\Phi(L_{1x}, y, z, t) &= \Phi(L_{2x}, y, z, t) = \Phi(x, 0, z, t) = \Phi(x, L_y, z, t) \\ &= \Phi(x, y, 0, t) = \Phi(x, y, L_z, t) = 0,\end{aligned} \tag{30}$$

which describes the moving boundary problem in three dimensions. The field operator in the Heisenberg representation $\Phi(x, y, z, t)$ obeys the wave equation $(c = 1)$ in three dimensions

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial t^2}. \tag{31}$$

We can rewrite this equation in the moving coordinate systems, (q, y, z). The transformation of the time derivative operator is given in (10). Hence, the wave equation is transformed to

$$\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{L^2}\frac{\partial^2 \Phi}{\partial q^2} = \frac{\partial^2 \Phi}{\partial t^2} - \epsilon\left(2\Omega \cos(\Omega t)\frac{\partial^2 \Phi}{\partial q \partial t} - \Omega^2 \sin(\Omega t)\frac{\partial \Phi}{\partial q}\right). \tag{32}$$

In one dimension, the expansion of the solution of the wave equation was given by formula (12). In three dimension, it has obvious generalization. The scalar field operator can be expanded as

$$\Phi(q, y, z, t) = \sum_{k_x,k_y,k_z} b_{k_xk_yk_z}\Psi_{k_xk_yk_z}(q, y, z, t) + \text{H.C.}, \tag{33}$$

where $b_{k_xk_yk_z}$ is the annihilation operator and $\Psi_{k_xk_yk_z}(q, y, z, t)$ is the corresponding mode function. Let us assume that, for an arbitrary moment of time, the explicit form of function $\Psi_{k_xk_yk_z}(q, y, z, t)$ is expanded as

$$\Psi_{k_xk_yk_z}(q, y, z, t > 0) = \sum_{n_x,n_y,n_z} a^{k_xk_yk_z}_{n_xn_yn_z}(t)\sin(n_x\pi q)\sin\left(\frac{n_y\pi}{L_y}y\right)\sin\left(\frac{n_z\pi}{L_z}z\right). \tag{34}$$

Substituting these into the transformed wave equation (32) and using the orthogonality relations, we get an infinite set of coupled differential equations for $a^{k_xk_yk_z}_{n_xn_yn_z}(t)$ after some algebra

$$\begin{aligned}\ddot{a}^{k_xk_yk_z}_{m_xn_yn_z} &+ \omega^2_{m_xn_yn_z}a^{k_xk_yk_z}_{m_xn_yn_z}\\ &= \epsilon\left(4\Omega\cos(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}\dot{a}^{k_xk_yk_z}_{n_xn_yn_z} - 2\Omega^2\sin(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}a^{k_xk_yk_z}_{n_xn_yn_z}\right),\end{aligned} \tag{35}$$

where $\omega^2_{m_xn_yn_z} = \pi^2\left(m_x^2/L_x^2 + n_y^2/L_y^2 + n_z^2/L_z^2\right)$ and $g_{n_xm_x} = \frac{m_xn_x(1-(-1)^{m_x+n_x})}{m_x^2-n_x^2}$ for $m_x \neq n_x$. In fact, this equation is the three-dimensional generalization of (14).

We will apply the multiple scale analysis to solve this equation. Fortunately, there is no need to go into the detail. The generalization of equation (18) to three dimensions is straightforward

$$\begin{aligned}\ddot{a}^{k_xk_yk_z(1)}_{m_xn_yn_z} + \omega^2_{m_xn_yn_z}a^{k_xk_yk_z(1)}_{m_xn_yn_z} &= -2\partial_\tau\partial_t a^{k_xk_yk_z(0)}_{m_xn_yn_z} + 4\Omega\cos(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}\dot{a}^{k_xk_yk_z(0)}_{n_xn_yn_z}\\ &\quad - 2\Omega^2\sin(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}a^{k_xk_yk_z(0)}_{n_xn_yn_z},\end{aligned} \tag{36}$$

where

$$a^{k_xk_yk_z(0)}_{m_xn_yn_z} = A^{k_xk_yk_z}_{m_xn_yn_z}(\tau)\exp(-\mathrm{i}\omega_{m_xn_yn_z}t) + B^{k_xk_yk_z}_{m_xn_yn_z}(\tau)\exp(\mathrm{i}\omega_{m_xn_yn_z}t), \tag{37}$$

$A^{k_xk_yk_z}_{m_xn_yn_z}(\tau)$ and $B^{k_xk_yk_z}_{m_xn_yn_z}(\tau)$ depend on the slow time parameter τ. The initial conditions are given by

$$\begin{aligned}A^{k_xk_yk_z}_{n_xn_yn_z}(\tau = 0) &= \frac{1}{\sqrt{2\omega_{k_xk_yk_z}}}\delta_{k_x,n_x}\delta_{k_y,n_y}\delta_{k_z,n_z};\\ B^{k_xk_yk_z}_{n_xn_yn_z}(\tau = 0) &= 0.\end{aligned} \tag{38}$$

We will derive an equation like (21), (23). Substituting (37) into (36) and eliminating the secular terms from the equation, we get

$$\begin{aligned}\partial_\tau A^{k_xk_yk_z}_{m_xn_yn_z} &+ G^-_{(n'_xn_yn_z),(m_xn_yn_z)}A^{k_xk_yk_z}_{n'_xn_yn_z} - G^+_{(n''_xn_yn_z),(m_xn_yn_z)}A^{k_xk_yk_z}_{n''_xn_yn_z}\\ &- G^-_{(n'''_xn_yn_z),(m_xn_yn_z)}B^{k_xk_yk_z}_{n'''_xn_yn_z} = 0\end{aligned} \tag{39}$$

$$\begin{aligned}-\partial_\tau B^{k_xk_yk_z}_{m_xn_yn_z} &- G^-_{(n'_xn_yn_z),(m_xn_yn_z)}B^{k_xk_yk_z}_{n'_xn_yn_z} + G^+_{(n''_xn_yn_z),(m_xn_yn_z)}B^{k_xk_yk_z}_{n''_xn_yn_z}\\ &+ G^-_{(n'''_xn_yn_z),(m_xn_yn_z)}A^{k_xk_yk_z}_{n'''_xn_yn_z} = 0,\end{aligned} \tag{40}$$

where the three-dimensional generalization of the definitions $G_{ij}^{\mp}$ (22) is given by

$$G^{\mp}_{(r_x,n_y,n_z),(m_x,n_y,n_z)} = \frac{\Omega \mp 2\omega_{r_x n_y n_z}}{2\omega_{m_x n_y n_z}}\,\Omega g_{r_x m_x}. \tag{41}$$

It should be noted that n_x', n_x'', n_x''' are positive integers and they satisfy

$$\begin{aligned} \omega_{n_x' n_y n_z} &= \Omega + \omega_{m_x n_y n_z},\\ \omega_{n_x'' n_y n_z} &= -\Omega + \omega_{m_x n_y n_z},\\ \omega_{n_x''' n_y n_z} &= \Omega - \omega_{m_x n_y n_z}. \end{aligned} \tag{42}$$

In the one-dimensional case, $A_m, A_{m\mp p}, A_{m\mp 2p}, \ldots$ and $B_m, B_{m\mp p}, B_{m\mp 2p}, \ldots$ are strongly coupled to each other. However, in three dimensions, only a few modes are coupled to each other. This is because there are only a few positive integers n_x', n_x'', n_x''' satisfied by equations (42). So, it is possible to solve equations (39), (40) exactly since only a few modes are coupled.

In what follows, we will give some specific examples.

4.1.1. Examples. For simplicity, assume that the cavity is cubic, $L_x = L_y = L_z$. We have a freedom to choose the coupled modes. For example, assume that inter-mode coupling occurs between (n_x''', n_y, n_z) and (m_x, n_y, n_z). So we can determine Ω from (42). As an example, we are interested in the following two modes: (1, 1, 1) and (2, 1, 1). Choose $\Omega = (\sqrt{3} + \sqrt{6})\pi/L_x$. Let us solve equations (39), (40) for these modes. Hence,

$$\partial_\tau A_{211}^{k_x k_y k_z} - G^{-}_{(1,1,1),(2,1,1)} B_{111}^{k_x k_y k_z} = 0; \qquad -\partial_\tau B_{211}^{k_x k_y k_z} + G^{-}_{(1,1,1),(2,1,1)} A_{111}^{k_x k_y k_z} = 0. \tag{43}$$

To solve the above differential equations, we also need the equations for $\partial_\tau A_{1,1,1}$ and $\partial_\tau B_{1,1,1}$. Using again (39), (40), we get

$$\partial_\tau A_{111}^{k_x k_y k_z} - G^{-}_{(2,1,1),(1,1,1)} B_{211}^{k_x k_y k_z} = 0; \qquad -\partial_\tau B_{111}^{k_x k_y k_z} + G^{-}_{(2,1,1),(1,1,1)} A_{211}^{k_x k_y k_z} = 0. \tag{44}$$

The solution of equations (43), (44) with the boundary conditions (38) can readily be found.

$$\begin{pmatrix} A_{111}^{111} \\ A_{211}^{211} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2\omega_{111}}}\cosh(\lambda\tau) \\ \frac{1}{\sqrt{2\omega_{211}}}\cosh(\lambda\tau) \end{pmatrix}; \qquad \begin{pmatrix} B_{111}^{211} \\ B_{211}^{111} \end{pmatrix} = \begin{pmatrix} \frac{G^{-}_{(2,1,1),(1,1,1)}}{\lambda\sqrt{2\omega_{211}}}\sinh(\lambda\tau) \\ \frac{G^{-}_{(1,1,1),(2,1,1)}}{\lambda\sqrt{2\omega_{111}}}\sinh(\lambda\tau) \end{pmatrix}, \tag{45}$$

where the constant λ is defined as $\lambda^2 = G^{-}_{(2,1,1),(1,1,1)} G^{-}_{(1,1,1),(2,1,1)}$. With the help of number operator (24)

$$\langle N_{n_x n_y n_z}\rangle = \sum_{k_x}\sum_{k_y}\sum_{k_z} 2\omega_{n_x n_y n_z}\left|B_{n_x n_y n_z}^{(k_x k_y k_z)}\right|^2 \tag{46}$$

we find the number of generated photons for each mode

$$\langle N_{1,1,1}\rangle = \langle N_{2,1,1}\rangle = \sinh^2(\lambda\tau). \tag{47}$$

4.2. Vector field

Here, we will study the dynamical Casimir effect for the vector field in three dimensions. Maxwell's equations describe electromagnetic waves as having two components, the electric field, $E(x, y, z)$, and the magnetic field, $H(x, y, z)$. Modes in a cavity resonator are said to be transverse. It is convenient to classify the fields as transverse magnetic (TM) or transverse electric (TE) according to whether E or H was transverse to the direction of oscillation.

We will study the dynamical Casimir effect for the transverse electric modes. Before applying our formalism, let us first write TE modes in static case. In the mode TE, for $t < 0$ the cavity is static, and each mode is given by

$$
\begin{aligned}
&A_x = 0\\
&A_y = \sum_{n_x,n_y,n_z} A_{0_y} \exp(-\mathrm{i}\omega_{n_xn_yn_z} t) \sin\left(\frac{n_x\pi x}{L_x}\right)\cos\left(\frac{n_y\pi y}{L_y}\right)\sin\left(\frac{n_z\pi z}{L_z}\right)\\
&A_z = \sum_{n_x,n_y,n_z} A_{0_z} \exp(-\mathrm{i}\omega_{n_xn_yn_z} t) \sin\left(\frac{n_x\pi x}{L_x}\right)\sin\left(\frac{n_y\pi y}{L_y}\right)\cos\left(\frac{n_z\pi z}{L_z}\right),
\end{aligned}
\tag{48}
$$

where n_x, n_y, n_z are positive integers. The constants A_{0_y} and A_{0_z} satisfy the coulomb gauge condition, $A_{0_y} n_y/L_y + A_{0_z} n_z/L_z = 0$.

At time $t = 0$, the rectangular cavity resonator starts to oscillate in the x-direction as a whole. The positions of the six walls are given by (29). Let us find the components of the field operator at any time. We will study with the moving coordinate systems (q, y, z). The field operator $A(q, y, z, t)$ associated with a vector potential satisfies the transformed three-dimensional wave equation $(c = 1)$

$$
\frac{1}{L_x^2}\frac{\partial^2\vec{A}}{\partial q^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2} = \frac{\partial^2\vec{A}}{\partial t^2} - \epsilon\left(2\Omega\cos\Omega t\frac{\partial^2\vec{A}}{\partial q\partial t} - \Omega^2\sin\Omega t\frac{\partial\vec{A}}{\partial q}\right).
\tag{49}
$$

When $t > 0$, the solution of the components of the field operator may be expanded in terms of the orthogonal basis functions.

$$
\begin{aligned}
&A_x = 0\\
&A_y = \sum_{n_x,n_y,n_z} A_{0_y} a_{n_xn_yn_z}^{k_xk_yk_z}(t) \sin(n_x\pi q)\cos\left(\frac{n_y\pi y}{L_y}\right)\sin\left(\frac{n_z\pi z}{L_z}\right),\\
&A_z = \sum_{n_x,n_y,n_z} A_{0_z} a_{n_xn_yn_z}^{k_xk_yk_z}(t) \sin(n_x\pi q)\sin\left(\frac{n_y\pi y}{L_y}\right)\cos\left(\frac{n_z\pi z}{L_z}\right),
\end{aligned}
\tag{50}
$$

where the time-dependent function $a_{n_xn_yn_z}^{k_xk_yk_z}(t)$ is to be determined later. Let us substitute equation (50) into the transformed wave equation (49). Using the orthogonality relations, we obtain the dynamical equations.

$$
\begin{aligned}
&\ddot{a}_{m_xn_yn_z}^{k_xk_yk_z} + \omega_{m_xn_yn_z}^2 a_{m_xn_yn_z}^{k_xk_yk_z}\\
&\quad = \epsilon\left(4\Omega\cos(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}\dot{a}_{n_xn_yn_z}^{k_xk_yk_z} - 2\Omega^2\sin(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x} a_{n_xn_yn_z}^{k_xk_yk_z}\right).
\end{aligned}
\tag{51}
$$

As can be seen, this equation and equation (35) for the scalar field are the same. So, for the dynamical Casimir effect problem, both scalar and vector field cases can be treated in the similar way. Multiple scale analysis has already been applied to the scalar field. So, equations (39), (40) are also valid for TE modes. We will study the photon generation for the TE modes by giving an example.

We demand that the generated photons have the modes (2, 2, 1) and (3, 2, 1). These two modes are coupled if we choose $\Omega = (3 + \sqrt{14})\pi/L_x$.

After performing the same calculations with the help of MSA we obtain,

$$
\begin{pmatrix} A_{221}^{221} \\ A_{321}^{321} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2\omega_{2,2,1}}}\cosh(\lambda'\tau) \\ \frac{1}{\sqrt{2\omega_{3,2,1}}}\cosh(\lambda'\tau) \end{pmatrix}; \qquad \begin{pmatrix} B_{221}^{221} \\ B_{321}^{321} \end{pmatrix} = \begin{pmatrix} \frac{G^{-}_{(3,2,1),(2,2,1)}}{\lambda'\sqrt{2\omega_{2,1,1}}}\sinh(\lambda'\tau) \\ \frac{G^{-}_{(2,2,1),(3,2,1)}}{\lambda'\sqrt{2\omega_{1,1,1}}}\sinh(\lambda'\tau) \end{pmatrix},
\tag{52}
$$

where $\lambda'^2 = G^-_{321,221} G^-_{221,321}$. Then, we calculate the number of generated photons (24)

$$\langle N_{2,2,1}\rangle = \langle N_{3,2,1}\rangle = \sinh^2(\lambda'\tau). \tag{53}$$

Here, we have performed analytical calculations to find the number of generated photons by using MSA. To this end, it should be mentioned that Ruser found perfect agreement between the numerical results and analytical predictions obtained by MSA [33]. The equations of motion for TE modes in a dynamical rectangular cavity are equivalent to the equations of motion for a scalar field with a time-dependent Dirichlet boundary conditions. More complicated boundary conditions, the so-called Neumann boundary conditions, arises when studying TM modes.

5. Enhancement of photon numbers

So far, we have considered the cavity resonator oscillated as a whole. We will now study the case of symmetric oscillation with respect to the center of the cavity in the x-direction.

$$\begin{aligned} &L_{1x}(t) = -\epsilon L_x \sin\Omega t, \qquad L_{2x}(t) = L_x(1+\epsilon\sin\Omega t), \qquad L_{1y}(t) = 0, \\ &L_{2y}(t) = L_y, \qquad L_{1z}(t) = 0, \qquad L_{2z}(t) = L_z. \end{aligned} \tag{54}$$

In this case, the volume of the cavity changes in time. The two walls in the x-direction move opposite to each other. For this configuration, only the scalar field will be treated. This is because, as was pointed above, the equations for $a^{k_xk_yk_z}_{n_xn_yn_z}(t)$ are the same for the scalar field and the vector field. In other words, the number of produced TE-mode photons equals the number of produced scalar particles in a three-dimensional cavity. So, it is enough to study the dynamical Casimir effect for the scalar field to understand the underlying mechanism. For an arbitrary moment of time, the mode function for the scalar field is expanded as

$$\Psi_{k_xk_yk_z}(t>0) = \sum_{n_xn_yn_z} a^{k_xk_yk_z}_{n_xn_yn_z}(t)\sqrt{\frac{L_x}{L_{2x}-L_{1x}}}\sin(n_x\pi q)\sin\left(\frac{n_y\pi}{L_y}y\right)\sin\left(\frac{n_z\pi}{L_z}z\right), \tag{55}$$

where $q(t) = \frac{x-L_{1x}}{L_{2x}-L_{1x}}$. If we substitute it into the wave equation and use the orthogonality relations, we get

$$\begin{aligned} &\ddot{a}^{k_xk_yk_z}_{m_xn_yn_z} + \omega^2_{m_xn_yn_z}(t)a^{k_xk_yk_z}_{m_xn_yn_z} \\ &\quad = \epsilon\left(4\Omega\cos(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}\dot{a}^{k_xk_yk_z}_{n_xn_yn_z} - 2\Omega^2\sin(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}a^{k_xk_yk_z}_{n_xn_yn_z}\right), \end{aligned} \tag{56}$$

where $\omega^2_{m_xn_yn_z}(t) = \pi^2\big(m_x^2/(L_{2x}-L_{1x})^2 + n_y^2/L_y^2 + n_z^2/L_z^2\big)$ and the new antisymmetric coefficient is given by $g_{n_xm_x} = \frac{m_xn_x(1+(-1)^{m_x+n_x})}{n_x^2-m_x^2}$ for $m_x \neq n_x$.

There are two differences between (35) and (56). First, the antisymmetric coefficient vanishes when m_x+n_x is an even number when the cavity oscillates as a whole (given below (35)). However, it vanishes when m_x+n_x is an odd number when the cavity oscillates symmetrically. Second, the term $\omega^2_{m_xn_yn_z}$ in the left-hand side of the former one is constant while it is time dependent for the latter one. The time-dependent character of it gives a modification of equation (36)

$$\begin{aligned} \ddot{a}^{k_xk_yk_z(1)}_{m_xn_yn_z} + \omega^2_{m_xn_yn_z}a^{k_xk_yk_z(1)}_{m_xn_yn_z} &= -2\partial_\tau\partial_t a^{k_xk_yk_z(0)}_{m_xn_yn_z} + 4\frac{\pi^2m_x^2}{L_x^2}\sin(\Omega t)a^{k_xk_yk_z(0)}_{m_xn_yn_z} \\ &\quad + 4\Omega\cos(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}\dot{a}^{k_xk_yk_z(0)}_{n_xn_yn_z} - 2\Omega^2\sin(\Omega t)\sum_{n_x\neq m_x} g_{n_xm_x}a^{k_xk_yk_z(0)}_{n_xn_yn_z}. \end{aligned} \tag{57}$$

The second term in the right-hand side is new. Let us study the the parametric resonance case, $\Omega = 2\omega_{m_x n_y n_z}$. Then, MSA gives equations for $A(\tau)$ and $B(\tau)$

$$\partial_\tau A_{m_x n_y n_z}^{k_x k_y k_z} + \frac{\pi^2 m_x^2}{L_x^2 \omega_{m_x}} B_{m_x n_y n_z}^{k_x k_y k_z} + G^-_{(n_x n_y n_z),(m_x n_y n_z)} A_{n_x n_y n_z}^{k_x k_y k_z} = 0 \tag{58}$$

$$-\partial_\tau B_{m_x n_y n_z}^{k_x k_y k_z} - \frac{\pi^2 m_x^2}{L_x^2 \omega_{m_x}} A_{m_x n_y n_z}^{k_x k_y k_z} - G^-_{(n_x n_y n_z),(m_x n_y n_z)} B_{n_x n_y n_z}^{k_x k_y k_z} = 0, \tag{59}$$

where $G^-_{(n_x n_y n_z),(m_x n_y n_z)}$ was defined in (41) and n_x is a positive integer which satisfy the following relation:

$$\omega_{n_x n_y n_z} = 3\omega_{m_x n_y n_z}. \tag{60}$$

Let us give some examples. First, we are interested in the uncoupled modes. For example, consider the mode (1, 1, 0). If we solve (58), (59) endowed with the initial conditions (38), we get

$$A_{110}^{110}(\tau) = \frac{1}{\sqrt{2\omega_{110}}} \cosh(\lambda\tau) \tag{61}$$

$$B_{110}^{110}(\tau) = -\frac{1}{\sqrt{2\omega_{110}}} \sinh(\lambda\tau), \tag{62}$$

where $\lambda = \frac{\pi}{\sqrt{2}L_x}$. The number of photons is calculated (24)

$$\langle N_{1,1,0}\rangle = \sinh^2(\lambda\tau). \tag{63}$$

Compare this result with the one obtained in [18] where the authors assumed that only one wall is oscillating in the parametric resonance case. They found that $\langle N_{1,1,0}\rangle = \sinh^2(\lambda_D\tau)$, where $\lambda_D = \lambda/2$.

The number of generated photons is the same for the following two systems: the cavity with a single oscillating mirror with 2ϵ and the cavity with two oppositely oscillating mirrors with ϵ. Instead of increasing ϵ by factor 2, the static wall is allowed to oscillate as described above.

As a second example consider the mode (1, 1, 1). In this case, according to (60), the mode (5, 1, 1) is coupled. The solutions of (58), (59) become

$$A_{511}^{511}(\tau) = \frac{1}{\sqrt{2\omega_{511}}} \delta_{5,k_x} \cosh(\lambda_1\tau) \tag{64}$$

$$B_{511}^{511}(\tau) = -\frac{1}{\sqrt{2\omega_{511}}} \delta_{5,k_x} \sinh(\lambda_1\tau), \tag{65}$$

$$\begin{aligned} A_{111}^{k_x 11}(\tau) &= 0.681 \frac{\delta_{k_x,5}}{\sqrt{2\omega_{511}}} (\sinh(\lambda_1\tau) - \sinh(\lambda_2\tau)) + \frac{\delta_{k_x,1}}{\sqrt{2\omega_{111}}} \cosh(\lambda_2\tau) \\ B_{111}^{k_x 11}(\tau) &= 0.681 \frac{\delta_{k_x,5}}{\sqrt{2\omega_{511}}} (\cosh(\lambda_2\tau) - \cosh(\lambda_1\tau)) - \frac{\delta_{k_x,1}}{\sqrt{2\omega_{111}}} \sinh(\lambda_2\tau), \end{aligned} \tag{66}$$

where $\lambda_1 = \frac{\pi}{L_x}\frac{25}{\sqrt{27}}$ and $\lambda_2 = \frac{\pi}{L_x}\frac{1}{\sqrt{3}}$. Then, the number of photons

$$\begin{aligned} \langle N_{1,1,1}\rangle &= \sinh^2(\lambda_2\tau) + 0.1549(\cosh^2(\lambda_2\tau) + \cosh^2(\lambda_1\tau) - 2\cosh^2(\lambda_2\tau)\cosh^2(\lambda_1\tau)) \\ \langle N_{5,1,1}\rangle &= \sinh^2(\lambda_1\tau). \end{aligned} \tag{67}$$

As a result, compared to the result obtained for a single mirror, the radiated photon flux is enhanced.

6. Discussion

Kim, Brownell and Onofrio proposed an experiment for the detection of the dynamical Casimir effect [13]. They considered a three-dimensional cavity with a single moving boundary. Taking into account the limitation by the photon leakage of the cavity expressed through the optical quality factor Q_{opt}, which saturates at the hold time $\tau = Q_{\text{opt}}/\omega$, they gave the formula for the maximum photon population for the parametric case and uncoupled mode as

$$\langle N \rangle = \sinh^2(2Q_{\text{opt}}\epsilon t). \tag{68}$$

We have shown that if the two walls are moving opposite to each other with the same frequencies and amplitudes, ϵ should be replaced by 2ϵ. Hence

$$\langle N \rangle = \sinh^2(4Q_{\text{opt}}\epsilon t). \tag{69}$$

The number of generated photons in the cavity is very sensitive to the product ϵQ_{opt}. In current technology, the maximum value $\epsilon Q_{\text{opt}} = 1$ [13]. The equation (68) gives the number of generated photons $N = 13$ if only one wall is in motion. If two walls move symmetrically with respect to the center of the cavity, (69) gives the number of generated photons $N = 745$.

The difference between the two cases is great if $\epsilon Q_{\text{opt}} = 2$, which may be possible in the future. In this case, the first formula (68) gives $N = 745$. However, the second one (69) gives a large number of photons $N = 2.210^6$.

From the experimental point of view, we think that the systems with two moving walls will play an important role for the detection of generated photons. It may turn out to be very difficult to make the two walls oscillate symmetrically at exactly the same frequencies. The case of slightly off resonant external frequency, i.e, $\Omega \rightarrow \Omega + h$, where the detuning frequency h is sufficiently small $h \ll \Omega$, was already considered in the literature [32, 34]. It was shown that exponential photon production can still be observed provided that certain threshold conditions are satisfied.

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